

## 10.95

**Quantization of massless fields****over the static Robertson-Walker space****of constant negative curvature**

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**Abstract.** Taking the  $\mathbb{R}^1 \times H^3$  space as an example, we develop the new method of quantization of fields over symmetric spaces. We construct the quantized massless fields of an arbitrary spin over the  $\mathbb{R}^1 \times H^3$  space by the resolution over the systems of "plane waves" which are solutions of the corresponding wave equations. The propagators of these fields are  $\mathbb{R}^1 \times SO(3,1)$ -invariant and causal. For spin 0 and 1/2 fields the propagators are obtained in the explicit form.

**1. Introduction**

As the Minkowski space is only an idealization of a real curved space, then the problem of construction of QFT over the curved space appears naturally. The solution of this problem shall be most complete and formally similar to that over the flat space if we restrict ourselves to the symmetric spaces with symmetry groups of a sufficiently high range. To this end in [1] we applied the method of generalized coherent states (CS) to the quantization of massive spin 0 and 1/2 fields over de Sitter space. Another curved space in which we can apply this method is the  $\mathbb{R}^1 \times H^3$  space. Like de Sitter space, it is the particular case of Robertson-Walker spaces of constant curvature; from the other hand, it belongs to the class of so-called ultrastatic spaces which is also important as an arena of QFT [2]. In [3] using the CS method we found the solutions of wave equations for the massless particles of arbitrary spin over the  $\mathbb{R}^1 \times H^3$  space in the form of so-called "plane waves", and using these "plane waves" for the spin zero particles we proposed a new method of quantization of spin zero massless field over this space. In the present paper we apply the CS method to the quantization of massless fields of arbitrary nonzero spin over this space. The previous attempts to construct the spinor propagators over the  $\mathbb{R}^1 \times S^N$  [4] and  $H^N$  [5] spaces concerned with the massive spin 1/2 fields only and did not yield the results of a clear physical and mathematical sense. General results on the quantization of massive spinor and vector fields over static space-times may be found in [6].

The present paper is constructed as follows. In section 2 we state the well known results concerning the Lorentz group and its irreducible representations. In section 3 we consider the scalar CS system for the Lorentz group which is a generalization of that considered in [7] onto the arbitrary values of parameter. In section 4 we consider the spinor CS system for the Lorentz group introduced implicitly by S.Weinberg in [8]. Following [3], we construct the system of solutions of wave equations for the massless particles of arbitrary spin over the  $\mathbb{R}^1 \times H^3$  space as the product of this CS system and that constructed in section 3, and find the invariance properties of this system of solutions. In section 5 we construct the quantized massless fields of arbitrary spin by the resolution over the system of solutions constructed in section 4, and show that these fields have the  $\mathbb{R}^1 \otimes SO(3,1)$ -invariant propagators, and find the explicit form of these propagators in the case of spin 0 and 1/2 fields. In Section 6 we compare our results with other approaches to the quantization of fields in curved spaces. In Appendix A the arbitrary spin two-point functions over  $H^3$  space are derived. In Appendix B the causality of propagators for an arbitrary spin constructed in Section 5 and analyticity of corresponding two-point functions are proved. In Appendix C we show that these propagators possess the correct Minkowski space limit.

**2. The  $H^3$  space and its symmetry group**

The three-dimensional hyperbolic space  $H^3$  is the hyperboloid of the radius  $R$  in a four-dimensional Lorentzian space with the metric tensor  $\eta_{\alpha\beta} = \text{diag } (+1, +1, +1, -1)$ ,  $\alpha, \beta = 1, \dots, 4$ , determined by the equation  $\eta_{\alpha\beta} x^\alpha x^\beta = -R^2$ . We denote  $\kappa x = \sqrt{1 + x^2/R^2}$ . The symmetry group of the  $H^3$  space is  $SO(3,1)$  with the generators  $J_{\mu\nu} = -J_{\nu\mu}$  and its commutation relations are

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho}.$$

For the finite-dimensional irreducible representations the function  $f(\mathbf{z}, \bar{\mathbf{z}})$  should be a polynomial on  $\mathbf{z}$  and  $\bar{\mathbf{z}}$  of the power equal or less than  $2j_+$  and  $2j_-$ , respectively. Hereafter we shall deal with the finite-dimensional representations  $(\mathbf{s}, \mathbf{0})$ ; their generators are

$$J_{i0} = \frac{i}{2} \varepsilon_{ikl} J_{kl} = S_i \quad (2.2)$$

where  $S_i$  are generators of the rotation subgroup:

$$S_+ = -\frac{\partial}{\partial z} \quad S_- = z^2 \frac{\partial}{\partial z} - 2sz \quad S_3 = s - z \frac{\partial}{\partial z}. \quad (2.3)$$

We denote as  $L_{\alpha\beta}(g)$  the matrix of orthogonal transformation corresponding to the element  $g \in SO(3, 1)$  and define the action of the Lorentz group over the  $H^3$  space as  $\mathbf{x}^\alpha \mapsto \mathbf{x}_g^\alpha = L_{\beta}^{\alpha}(g) \mathbf{x}^\beta$ . The stationary subgroup of an arbitrary point of the  $H^3$  space is  $SO(3)$ ; then we can identify this space with the coset space  $SO(3, 1)/SO(3)$ .

### 3. Scalar coherent states

Let us consider an arbitrary light-like four-vector:

$$n^\alpha = (\omega \mathbf{q}, \omega) \quad \mathbf{q}^2 = 1. \quad (3.1)$$

Define the orthogonal action of the Lorentz group on it:

$$n^\alpha \mapsto n_g^\alpha = L_{\beta}^{\alpha}(g) n^\beta$$

and represent again the resulting vector in the form of (3.1). This yields the projective action of the Lorentz group over the unit three-vector:

$$\mathbf{q} \mapsto \mathbf{q}_g.$$

Let us consider the representation of the Lorentz group acting over the functions depending on  $\mathbf{q}$ :

$$T_\sigma(g) f(\mathbf{q}) = \left( \frac{n_g^4}{n^4} \right)^\sigma f(\mathbf{q}_{g^{-1}}) \quad (3.2)$$

where  $\sigma \in \mathbb{C}$ . At  $\sigma = i\omega R - 1$  the above representation is the infinite-dimensional and irreducible one for the spin zero particles. In [7] was shown that in this case the corresponding CS system for the  $H^3$  space is

$$\phi_{\mathbf{q}}(\mathbf{x}; \sigma) = \left( \kappa \mathbf{x} - \frac{\mathbf{q}\mathbf{x}}{R} \right)^\sigma.$$

This result may be proved for an arbitrary  $\sigma \in \mathbb{C}$  in the quite similar way. Then (3.2) yields the following transformation property of the constructed CS system for an arbitrary  $\sigma$ :

$$\phi_{\mathbf{q}}(\mathbf{x}_g; \sigma) = \left( \frac{n_g^4}{n^4} \right)^\sigma \phi_{\mathbf{q}'}(\mathbf{x}; \sigma) \quad \mathbf{q}' = \mathbf{q}_{g^{-1}}. \quad (3.3)$$

It is easily seen [3] that the functions

$$\psi(\mathbf{x}, t) = e^{i\omega t} \phi_{\mathbf{q}}(\mathbf{x}; i\omega R - 1) \quad (3.4)$$

obey the Klein-Gordon equation for the conformally-coupled massless field over the  $\mathbb{R}^1 \times H^3$  space. Computing the Jacobian of the transformation from  $\mathbf{q}$  to  $\mathbf{q}'$  and using (3.3) it is easy to show that the two-point function

$$\mathcal{W}^{(0)}(\mathbf{x}, \mathbf{y}; \omega) = \int_{S^2} d^2 q \phi_{\mathbf{q}}(\mathbf{x}; i\omega R - 1) \phi_{\mathbf{q}}(\mathbf{y}; -i\omega R - 1)$$

is  $SO(3, 1)$ -invariant:

$$\mathcal{W}^{(0)}(\mathbf{x}_g, \mathbf{y}_g; \omega) = \mathcal{W}^{(0)}(\mathbf{x}, \mathbf{y}; \omega).$$

Using (A.3) we can write it the explicit form:

$$\mathcal{W}^{(0)}(\mathcal{X}, \mathcal{Y}; \omega) = \frac{4\pi \sin \omega R \alpha(\mathcal{X}, \mathcal{Y})}{\omega R \sinh \alpha(\mathcal{X}, \mathcal{Y})} \quad (3.5)$$

where  $\alpha(\mathcal{X}, \mathcal{Y})$  is the geodesic distance between the points:  $\cosh \alpha(\mathcal{X}, \mathcal{Y}) = -R^{-2} \mathcal{X} \alpha \mathcal{Y}^\alpha$ . The above expression coincides with the spherical function for the spin-zero infinite-dimensional representations of the Lorentz group obtained in [9].

The equality

$$\frac{1}{(2\pi)^3} \int \frac{d^3 \mathcal{X}}{\kappa \mathcal{X}} \phi_{\mathbf{Q}}(\mathcal{X}; i\omega R - 1) \phi_{\mathbf{Q}'}(\mathcal{X}; -i\omega' R - 1) = \frac{1}{4} \delta(\mathbf{n} - \mathbf{n}') \quad (3.6)$$

holds [10].

#### 4. Spinor coherent states

Let  $\mathcal{H}$  be a little Lorentz group of the vector  $\mathbf{n}^\alpha = (0, 0, 1, 1)$ . It is easy to show that its generators are  $J_{10} + J_{13}$ ,  $J_{20} + J_{23}$  and  $J_{12}$ . Denote the three-dimensional rotation around the axis lying in the  $\mathcal{X}\mathcal{Y}$  plane as  $g_{\mathcal{X}\mathcal{Y}}$  and parametrize these rotations by the three-vector  $\mathbf{Q}$  which is the result of action of  $g_{\mathcal{X}\mathcal{Y}}$  onto the standard three-vector  $(0, 0, 1)$ . Expression for the  $g_{\mathcal{X}\mathcal{Y}}(\mathbf{Q})$  in the  $SL(2, \mathbb{C})$ -form is [11]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{2(1+q^3)}} \begin{pmatrix} 1+q^3 & q^1 - iq^2 \\ -q^1 - iq^2 & 1+q^3 \end{pmatrix}.$$

Denote the boost along the third axis transforming the four-vector  $(0, 0, 1, 1)$  into the vector  $(0, 0, \omega, \omega)$  as  $h_z = h_z(\omega)$ ;  $SL(2, \mathbb{C})$ -form of this transformation is

$$h_z(\omega) = \begin{pmatrix} e^{\omega/2} & 0 \\ 0 & e^{-\omega/2} \end{pmatrix}.$$

Then the transformation  $g_{\mathcal{X}\mathcal{Y}}(\mathbf{Q})h_z(\omega)$  transforms the vector  $(0, 0, 1, 1)$  into the vector (3.1) and then the mapping

$$\mathbf{n}^\alpha = (\omega \mathbf{Q}, \omega) \mapsto g_{\mathcal{X}\mathcal{Y}}(\mathbf{Q})h_z(\omega) \in SO(3, 1)$$

composes the lifting from the  $SO(3, 1)/\mathcal{H}$  space to the Lorentz group. From (2.2), (2.3) it follows that in the finite-dimensional representation  $(j_+, j_-)$  the vector, being  $\mathcal{H}$ -invariant to within a phase multiplier, is  $|\psi_0\rangle = 1$ . Then we can construct the CS system

$$|\mathbf{Q}\omega; j_+ j_-\rangle = T_{j_+ j_-}(g_{\mathcal{X}\mathcal{Y}}(\mathbf{Q})h_z(\omega))|\psi_0\rangle. \quad (4.1)$$

Using (2.1) it is easily seen that

$$|\mathbf{Q}\omega; j_+ j_-\rangle = \left( \frac{n^4 + n^3}{2} \right)^{j_+ + j_-} (1 - z\rho\mathbf{Q})^{2j_+} (1 - \bar{z}\bar{\rho}\mathbf{Q})^{2j_-} \quad (4.2)$$

where  $\rho\mathbf{Q} = \frac{q^1 + iq^2}{1+q^3}$ . The transformation property of the constructed CS system is

$$T_{j_+ j_-}(g)|\mathbf{Q}\omega; j_+ j_-\rangle \sim \left( \frac{n_g^4}{n^4} \right)^{j_+ + j_-} |\mathbf{Q}'\omega; j_+ j_-\rangle \quad \mathbf{Q}' = \mathbf{Q}_{g^{-1}} \quad (4.3)$$

where the equivalency relation  $\sim$  is the equality to within a phase multiplier. Indeed, the transformation  $T_{j_+ j_-}(g)$  transforms the vector  $|\mathbf{Q}\omega; j_+ j_-\rangle$  into the vector equivalent to  $|\mathbf{Q}'\omega'; j_+ j_-\rangle$ . Since the vectors  $|\mathbf{Q}\omega; j_+ j_-\rangle$  depend on  $\omega$  as  $\omega^{j_+ + j_-}$  and  $\omega = n^4$ , then from here the equality (4.3) follows.

In fact, the CS system (4.2) was implicitly constructed as basic spinors for the finite-dimensional representations of the Lorentz group in the classical S.Weinberg's paper [8]. In this paper the basic spinors were determined by the relation equivalent to (4.1), and it was shown that the vector

$|\psi_0\rangle$  should be stable under the little Lorentz group to within a phase multiplier. In [11] the realization (2.1) of the Lorentz group irreducible representations was used to the explicit construction of the vectors  $T_{j_+j_-}(gxy(\mathbf{q}))|\psi_0\rangle$ .

Now we construct the functions

$$f_{\mathbf{q}\omega}^{(s)}(\mathcal{X}) = \phi_{\mathbf{q}}(\mathcal{X}; i\omega R - s - 1) |\mathbf{q}\omega; s0\rangle. \quad (4.4)$$

Then we can show [3] that the functions

$$\psi(\mathcal{X}, t) = e^{i\omega t} f_{\mathbf{q}\omega}^{(s)}(\mathcal{X})$$

obey the massless wave equations for arbitrary spin  $s$ . Using (4.3) and (3.3) we obtain the transformation properties

$$f_{\mathbf{q}\omega}^{(s)}(\mathcal{X}_g) \sim \left(\frac{n_g^4}{n^4}\right)^{i\omega R - 1} T_{s0}(g) f_{\mathbf{q}'\omega}^{(s)}(\mathcal{X}) \quad \mathbf{q}' = \mathbf{q}_{g^{-1}}.$$

Then the two-point function

$$\mathcal{W}^{(s)}(\mathcal{X}, \mathcal{Y}; \omega) = \int_{S^2} d^2q f_{\mathbf{q}\omega}^{(s)}(\mathcal{X}) \otimes \left(f_{\mathbf{q}\omega}^{(s)}(\mathcal{Y})\right)^\dagger$$

is  $SO(3, 1)$ -invariant:

$$\mathcal{W}^{(s)}(\mathcal{X}_g, \mathcal{Y}_g; \omega) = T_{s0}(g) \mathcal{W}^{(s)}(\mathcal{X}, \mathcal{Y}; \omega) T_{s0}^\dagger(g).$$

## 5. Quantized fields

Now we consider the quantized massless fields of arbitrary spin  $s$  over the  $\mathbb{R}^1 \times H^3$  space:

$$\begin{aligned} \Psi^{(s)}(x) &= \frac{2^{s-1/2}}{(2\pi)^{3/2}} \int_0^\infty \omega d\omega \int_{S^2} d^2q \\ &\times \left( C_s(\omega) e^{i\omega x^0} f_{\mathbf{q}\omega}^{(s)}(\mathcal{X}) a^{(+)}(\mathbf{n}; s) + C_s(-\omega) e^{-i\omega x^0} f_{\mathbf{q}, -\omega}^{(s)}(\mathcal{X}) a^{(-)\dagger}(\mathbf{n}; s) \right) \end{aligned}$$

where the four-vector  $\mathbf{n}^\alpha$  is given by (3.1), the creation-annihilation operators obey the nonvanishing (anti)commutation relations

$$[a^{(+)}(\mathbf{n}; s), a^{(+)\dagger}(\mathbf{n}'; s)]_\pm = [a^{(-)}(\mathbf{n}; s), a^{(-)\dagger}(\mathbf{n}'; s)]_\pm = n^4 \delta^3(\mathbf{n} - \mathbf{n}')$$

$C_0(\omega) = 1$  and

$$C_s(\omega) = (1 + (2i\omega R)^{-1})(1 + 3(2i\omega R)^{-1}) \dots (1 + s(i\omega R)^{-1})$$

at  $s = 1/2, 3/2, \dots$  and

$$C_s(\omega) = (1 + (i\omega R)^{-1})(1 + 2(i\omega R)^{-1}) \dots (1 + s(i\omega R)^{-1})$$

at  $s = 1, 2, \dots$ . Then the matrix elements of corresponding propagators in the basis (A.1) are

$$D_{lm}^{(s)}(x, y) \equiv [\Psi_l^{(s)}(x), \Psi_m^{(s)\dagger}(y)]_\pm = \frac{2^{2s-1}}{(2\pi)^3} \int_{-\infty}^\infty \omega d\omega |C_s(\omega)|^2 e^{i\omega(x^0 - y^0)} \mathcal{W}_{lm}^{(s)}(\mathcal{X}, \mathcal{Y}; \omega). \quad (5.1)$$

The above propagators are obviously invariant under the time translations and spatial  $SO(3, 1)$ -transformations. In Appendix B we show that for an arbitrary spin they obey the causality principle i.e. they are equal to zero at  $(x^0 - y^0)^2 - R^2(\alpha(x, y))^2 \neq 0$ . In this attitude these propagators are similar to those for massless fields of arbitrary spin over the Anti-de Sitter space constructed in [12]. In Appendix C we show that at  $R \rightarrow \infty$  these propagators pass onto the usual massless arbitrary spin propagators over Minkowski space constructed in [8].

Using (3.5) for the spin zero propagator we get [3]

$$D^{(0)}(\mathbf{x}, t_1; \mathbf{y}, t_1 + t) = -\frac{i\alpha}{2\pi \sinh \alpha} \delta(t^2 - R^2 \alpha^2) \epsilon(t). \quad (5.2)$$

The above expression coincides with the difference of the positive- and negative-frequency Wightmann functions, obtained previously by the standard methods [13], to within the multiple 1/4.

Using the equalities (A.4) we obtain the spin 1/2 two-point function

$$\begin{aligned} \mathcal{W}^{(1/2)}(\mathbf{x}, \mathbf{y}, \omega) &= i(C_{1/2}(\omega))^{-1} (\kappa \mathbf{x} + R^{-1} \boldsymbol{\sigma} \mathbf{x}) \\ &\times \left( \frac{1}{2R} - i\omega + \sigma_i e_{(i)}^k \frac{\partial}{\partial x^k} \right) \mathcal{W}^{(0)} \left( \mathbf{x}, \mathbf{y}, \omega - \frac{i}{2R} \right). \end{aligned}$$

The corresponding propagator may be represented in the form

$$D^{(1/2)}(x, y) = -2ie^{t/2R} (\kappa \mathbf{x} + R^{-1} \boldsymbol{\sigma} \mathbf{x}) \left( \frac{\partial}{\partial x^0} - \sigma_i e_{(i)}^k \frac{\partial}{\partial x^k} \right) D^{(0)}(x, y). \quad (5.3)$$

At  $R \rightarrow \infty$  the propagators (5.2), (5.3) pass onto the usual massless spin 0 and 1/2 propagators over Minkowski space constructed in [8].

## 6. Discussion

The common scheme of construction of quantum fields in curved space-time is the following [13]. Let  $I, J$  be the indices denoting the set of quantum numbers which distinguishes the states of our particles; these indices may run both discrete and continuous sets of values. Define two sets of creation-annihilation operators with nonvanishing (anti)commutation relations

$$[a_I, a_J^\dagger]_\pm = [b_I, b_J^\dagger]_\pm = \delta_{IJ}.$$

Introduce two sets  $\varphi_I^{(\pm)}(\mathbf{x})$  of solutions of corresponding relativistic wave equations; these solutions should be orthonormal with respect to the appropriate scalar product:

$$(\varphi_I^{(+)}, \varphi_J^{(+)}) = (\varphi_I^{(-)}, \varphi_J^{(-)}) = \delta_{IJ} \quad (6.1)$$

$$(\varphi_I^{(+)}, \varphi_J^{(-)}) = 0 \quad (6.2)$$

and should be of positive (negative) frequency with respect to the time-like Killing vector  $\xi$ :

$$\mathcal{L}_\xi \varphi_I^{(\pm)} = \pm i\omega_I \varphi_I^{(\pm)}. \quad (6.3)$$

Equality (6.1) means the separation between the particles with different quantum numbers, and the equalities (6.2), (6.3) mean the separation between particles and antiparticles. Then we can define the quantized field as

$$\Phi(x) = \sum_I (\varphi_I^{(+)}(x) a_I + \varphi_I^{(-)}(x) b_I^\dagger) \quad (6.4)$$

and its propagator is equal to

$$D(x, y) \equiv [\Phi(x), \Phi^\dagger(y)]_\pm = \sum_I (\varphi_I^{(+)}(x) \varphi_I^{(+)\dagger}(y) - \varphi_I^{(-)}(x) \varphi_I^{(-)\dagger}(y)). \quad (6.5)$$

However, the area of applicability of such a scheme turns to be limited. Usually it is impossible to perform the summation/integration at the r.h.s of (6.5) in the closed form, especially for the fields of nonzero spin. In the cases when it is possible, the resulting propagator may violate the causality principle, as in the case of massive spin 1/2, 1 and 2 fields over de Sitter space [14]. From the other hand, for the spin zero massless minimally coupled field over the same space it is necessary to introduce the noninvariant vacuum [15] or states with negative norm [16].

Another serious difficulty comes from the notion of particles. Generally speaking, it have not the invariant (independent from an observes) sense. Even in the cases when the unambiguous separation between the positive- and negative-frequency modes is possible (e.g. in the stationary space-times), the correct definition of particles may be still unavailable, as its takes place inside the Schwarzschild black hole [17]. Even in the space  $\mathbb{R}^1 \times H^3$  whose metrics is stationary and regular everywhere, the original pathology appears: the set of functions which obey (6.1) at finite  $R$ , does not obey this equality at  $R \rightarrow \infty$  due to the multiplier  $1/4$  at the r.h.s. of (3.6). Then the particles which are defined in the  $\mathbb{R}^1 \times H^3$  space due to (6.1), (6.2) and (6.3), do not coincide with the usual particles over Minkowski space at  $R \rightarrow \infty$ , in spite of the fact that the plane waves (3.4) in this limit pass into the usual plane waves over Minkowski space.

Then, constructing the QFT in curved space there is no reasons to start from the definition of particles given by (6.1), (6.2) and (6.3). However, in some important cases such as de Sitter and  $\mathbb{R}^1 \times H^3$  symmetric spaces, we can use another generalization of Minkowskian notion of particles via appropriate generalization of the Poincaré invariance. Namely, we can construct the functions  $\varphi_I^{(\pm)}$  as the "plane waves" which reflect the symmetry properties of the space and generalize the usual plane waves over Minkowski space. Such an approach is developed in [18, 1] for the massive spin 0, 1/2 and 1 fields over de Sitter space and in [3] and in the present paper for the massless arbitrary spin fields over the  $\mathbb{R}^1 \times H^3$  space. In [1, 3] and in the present paper we show that the CS method is a natural mathematical tool for constructing these "plane waves" and studying their properties.

In this approach the physical sensibility of the theory is based on the conditions imposed on the two-point functions

$$D^+(\mathbf{x}, \mathbf{y}) = \langle 0 | \Phi(\mathbf{x}) \Phi^\dagger(\mathbf{y}) | 0 \rangle = \sum_I \varphi_I^{(+)}(\mathbf{x}) \varphi_I^{(+)\dagger}(\mathbf{y})$$

rather on the functions  $\varphi_I^{(\pm)}$  itself, where  $\Phi(\mathbf{x})$  is still given by (6.4) and  $\mathbf{a}_I | 0 \rangle = \mathbf{b}_I | 0 \rangle = 0$ . In the case of the  $\mathbb{R}^1 \times H^3$  space the function  $D^+(\mathbf{x}, \mathbf{y})$  is the positive frequency part of the corresponding propagator.

Like (6.1), (6.2) and (6.3), the mentioned conditions generalize the corresponding Minkowski space conditions:

1) Invariance:

$$D^+(\mathbf{x}_g, \mathbf{y}_g) = U(g) D^+(\mathbf{x}, \mathbf{y}) \bar{U}(g)$$

where  $g \in \mathcal{G}$  is an arbitrary element of the symmetry group  $\mathcal{G}$  of our symmetric space  $X$ ;  $X \ni \mathbf{x} \mapsto \mathbf{x}_g \in X$  is the action of  $\mathcal{G}$  over this space, and  $U(g)$  is some finite-dimensional representation of  $\mathcal{G}$ .

2) Causality:  $D^+(\mathbf{x}, \mathbf{y}) = D^+(\mathbf{y}, \mathbf{x})$  if the points  $\mathbf{x}, \mathbf{y}$  are spacelike-separated (massive case) or not lightlike-separated (massless case).

3) Positiveness:

$$\int_X \int_X d\sigma(\mathbf{x}) d\sigma(\mathbf{y}) f(\mathbf{x}) D^+(\mathbf{x}, \mathbf{y}) f^*(\mathbf{y}) \geq 0$$

where  $d\sigma(\mathbf{x})$  is the  $\mathcal{G}$ -invariant measure on  $X$  and  $f(\mathbf{x})$  is an arbitrary  $C^\infty$ -function with compact support in  $X$ .

4) Analyticity:  $D^+(\mathbf{x}, \mathbf{y})$  is a boundary value of some analytic function defined in the certain domain of the appropriate complexification of  $X$ .

Then the Hilbert space structure of field theory may be obtained via the reconstruction theorem [19]. In [18, 1] it was shown that the above properties hold for the massive spin 0, 1/2 and 1 fields over de Sitter space. Within the approach advocated in [1, 3] and in the present paper, the properties 1) and 3) immediately follow from the construction of  $D^+(\mathbf{x}, \mathbf{y})$ . In Appendix B we show that the properties 2) and 4) also hold for the massless arbitrary spin fields over the  $\mathbb{R}^1 \times H^3$  space. Thus, the theory of these fields constructed above obeys all the requirements to the meaningful free QFT in curved space.

## Appendix A. Two-point functions

The polynomials

$$|m\rangle = (C_{2s}^{m-1})^{1/2} z^{m-1} \quad m = 1, \dots, 2s+1 \quad (\text{A.1})$$

compose the orthonormal basis in the spin  $s$  representation of the  $SO(3)$  group. We can consider them as the columns:

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \dots$$

We can obtain the arbitrary spin two-point function for the points

$$x_0^\alpha = (0, 0, R \sinh \alpha, R \cosh \alpha) \quad y_0^\alpha = (\mathbf{0}, R).$$

Then passing to the spherical coordinates over the 2-sphere and using the formula [20]

$${}_2F_1(a, b; c; z) = \frac{2^{1-c} \Gamma(c)}{\Gamma(b) \Gamma(c-b)} \times \int_0^\pi d\varphi \frac{(\sin \varphi)^{2b-1} (1 + \cos \varphi)^{c-2b}}{\left(1 - \frac{z}{2} + \frac{z}{2} \cos \varphi\right)^a}$$

we obtain

$$\mathcal{W}_{lm}^{(s)}(\mathcal{X}_0, \mathcal{Y}_0; \omega) = \frac{4\pi\omega^{2s}}{2s+1} e^{(-i\omega R + s+1)\alpha} \delta_{lm} {}_2F_1(-i\omega R + s+1, l; 2s+2; 1 - e^{2\alpha}). \quad (\text{A.2})$$

In [7] for the scalar product of two spin zero CS the expression

$$\mathcal{W}^{(0)}(\mathcal{X}, \mathcal{Y}; \omega) = 4\pi {}_2F_1\left(-\frac{i\omega R - 1}{2}, \frac{i\omega R + 1}{2}; \frac{3}{2}; -\sinh^2 \alpha(x, y)\right)$$

was obtained. It is a particular case of (A.2) since the equality [20]

$${}_2F_1(a, b; 2b; z) = (1-z)^{-a/2} {}_2F_1\left(\frac{a}{2}, b - \frac{a}{2}; b + \frac{1}{2}; \frac{z^2}{4(z-1)}\right)$$

holds. At  $s = 0$

$${}_2F_1(-i\omega R + 1, 1; 2; 1 - e^{2\alpha}) = \frac{e^{2i\omega R\alpha} - 1}{i\omega R(e^{2\alpha} - 1)}. \quad (\text{A.3})$$

At  $s = 1/2$

$$\begin{aligned} {}_2F_1\left(-i\omega R + \frac{3}{2}, 1; 3; 1 - e^{2\alpha}\right) &= -2 \frac{e^{2i\omega R\alpha} e^\alpha + \left(i\omega R + \frac{1}{2}\right)(1 - e^{2\alpha}) - 1}{\left(\omega^2 R^2 + \frac{1}{4}\right)(1 - e^{2\alpha})^2} \\ {}_2F_1\left(-i\omega R + \frac{3}{2}, 2; 3; 1 - e^{2\alpha}\right) &= 2 \frac{e^{2i\omega R\alpha} e^\alpha \left(i\omega R - \frac{1}{2}\right) - \left(i\omega R + \frac{1}{2}\right)e^{-\alpha} + 1}{\left(\omega^2 R^2 + \frac{1}{4}\right)(1 - e^{2\alpha})^2}. \end{aligned} \quad (\text{A.4})$$

## Appendix B. Causality

We denote

$$K_l^{(s)}(\alpha, \omega) = \omega |C_s(\omega)|^2 \mathcal{W}_l^{(s)}(\mathcal{X}_0, \mathcal{Y}_0; \omega).$$

Twice applying the expression [20]

$$\frac{(-1)^n (a)_n (c-b)_n}{(c)_n} (1-z)^{a-1} {}_2F_1(a+n, b; c+n; z) = \frac{d^n}{dz^n} \left( (1-z)^{a+n-1} {}_2F_1(a, b; c; z) \right)$$

where  $(a)_n = \Gamma(a+n)/\Gamma(a)$  and using the hypergeometric equation we obtain

$$\begin{aligned} & \frac{a(c-a)b(c-b)}{c(c+1)} {}_2F_1(a+1, b+1; c+2; z) \\ &= \left( \frac{d^2}{dz^2} + (c-a-b-1) \frac{d}{dz} \right) {}_2F_1(a, b; c; z). \end{aligned} \quad (\text{B.1})$$

Then

$$K_l^{(s)}(\alpha, \omega) = p_{s,l}(\alpha, \omega) e^{i\omega R\alpha} + q_{s,l}(\alpha, \omega) e^{-i\omega R\alpha} \quad (\text{B.2})$$

where  $p_{s,l}(\alpha, \omega)$  and  $q_{s,l}(\alpha, \omega)$  are the polynomials on  $\omega$ . Indeed, the validity of (B.2) at  $s = 0$  and  $s = 1/2$  follows from (A.3) and (A.4) immediately. Let (B.2) be valid for  $K_l^{(s)}$ ,  $l = 1, \dots, 2s + 1$ . Then its validity for  $K_l^{(s+1)}$  at  $l = 2, \dots, 2s + 2$  follows from (B.1). From the other hand, the validity of (B.2) for  $K_1^{(s+1)}$  and  $K_{2s+3}^{(s+1)}$  follows from its validity for  $K_2^{(s+1)}$  and  $K_{2s+2}^{(s+1)}$  using the expressions [20]

$$\begin{aligned} (a)_n z^{a-1} {}_2F_1(a+n, b; c; z) &= \frac{d^n}{dz^n} \left( z^{a+n-1} {}_2F_1(a, b; c; z) \right) \\ (c-a)_n z^{c-a-1} (1-z)^{a+b-c-n} {}_2F_1(a-n, b; c; z) \\ &= \frac{d^n}{dz^n} \left( z^{c-a+n-1} (1-z)^{a+b-c} {}_2F_1(a, b; c; z) \right). \end{aligned}$$

Thus (B.2) is proved for the arbitrary  $s$  and  $l$ . Then for the propagator we obtain

$$\begin{aligned} D_{lm}^{(s)}(t_1, \mathbf{x}_0; t_1 + t, \mathbf{y}_0) &= \delta_{lm} \frac{2^{2s+2} \pi^2}{2s+1} \\ &\times \left[ p_{s,l} \left( \alpha, -i \frac{\partial}{\partial(t+R\alpha)} \right) \delta(t+R\alpha) + q_{s,l} \left( \alpha, -i \frac{\partial}{\partial(t-R\alpha)} \right) \delta(t-R\alpha) \right]. \end{aligned} \quad (\text{B.3})$$

where we consider the differential operators in the arguments of  $p_{s,l}$  and  $q_{s,l}$  do not acting onto the first arguments of these functions. Then the propagator is not equal to zero only at  $|t| = R\alpha(\mathbf{x}_0, \mathbf{y}_0)$ ; then its causality for the arbitrary  $\mathbf{x}$  and  $\mathbf{y}$  follows from its Lorentz-invariance.

From (A.3), (A.4) and (B.1) it follows that the functions  $p_{s,l}(\alpha, \omega)$  and  $q_{s,l}(\alpha, \omega)$  have not singularities at  $\alpha \neq 0$ . Consider the domain

$$\{(\mathbf{x}, \mathbf{y}) | \Im(\mathbf{x}^0 - \mathbf{y}^0) > 0, \mathbf{x}, \mathbf{y} \in \mathbb{R}^3, \alpha(\mathbf{x}, \mathbf{y}) \neq 0\}. \quad (\text{B.4})$$

Since the positive-frequency part of the function  $\delta(\mathbf{x})$  is a boundary value of the function  $i\pi^{-1}\mathbf{x}^{-1}$  at  $\Im\mathbf{x} \rightarrow +0$ , then the two-point functions  $D^{(s)}(\mathbf{x}, \mathbf{y})$  are analytic in the domain (B.4), and they may be obtained from the r.h.s. of (B.3) replacing  $\delta(t \pm R\alpha)$  to  $i\pi^{-1}(t \pm R\alpha)^{-1}$ .

### Appendix C. Minkowski space limit

Since the functions (4.4) at  $R \rightarrow \infty$  pass onto the usual plane waves over Minkowski space constructed in [8, 11], then we can expect that the analogous passage takes place for the propagators too. However, since the corresponding integrals makes sense in the terms of generalized functions, then this limiting passage demands additional justification (cf. discussion of equation (3.6) in Section 6). Define

$$\tilde{D}^{(s)\text{flat}}(\mathbf{x} - \mathbf{y}) = \lim_{R \rightarrow \infty} D^{(s)}(\mathbf{x}, \mathbf{y})$$

where  $D^{(s)}(\mathbf{x}, \mathbf{y})$  is given by (5.1). The propagators for the arbitrary spin massless fields over Minkowski space are [8]

$$D^{(s)\text{flat}}(\mathbf{x}) = \frac{2^{2s-1}}{(2\pi)^3} \int_{n^2=0} \frac{d^3 n}{n^0} e^{in \cdot \mathbf{x}} |\mathbf{q}\omega; s0\rangle \langle \mathbf{q}\omega; s0|.$$

Now we prove that  $\tilde{D}^{(s)\text{flat}}(\mathbf{x}) = D^{(s)\text{flat}}(\mathbf{x})$ . At  $s = 0$  and  $s = 1/2$  it was proved in Section 5. Then we only should prove that the matrix elements of propagators  $\tilde{D}^{(s)\text{flat}}(\mathbf{x})$  and  $\tilde{D}^{(s+1)\text{flat}}(\mathbf{x})$  are connected with each other by the same recurrence relations as the matrix elements of  $D^{(s)\text{flat}}(\mathbf{x})$  and  $D^{(s+1)\text{flat}}(\mathbf{x})$  are. To this end we put  $\mathbf{x}^1 = \mathbf{x}^2 = 0$ . Then in the basis (A.1) we obtain

$$D_{lm}^{(s)\text{flat}}(\mathbf{x}) = \frac{C_{2s}^{m-1}}{2(2\pi)^3} \delta_{lm} \int_{n^2=0} \frac{d^3 n}{n^0} e^{in \cdot \mathbf{x}} (n^0 - n^3)^{2s+1-m} (n^0 + n^3)^{m-1}.$$

From the above expression the recurrence relations

$$\begin{aligned} (\partial_0 - \partial_3)^2 D_{mm}^{(s)\text{flat}}(\mathbf{x}) &= -(2s+2-m)(2s+3-m) D_{mm}^{(s+1)\text{flat}}(\mathbf{x}) \\ (\partial_0^2 - \partial_3^2) D_{2s+1 \ 2s+1}^{(s)\text{flat}}(\mathbf{x}) &= -\frac{1}{2s+2} D_{2s+2 \ 2s+2}^{(s+1)\text{flat}}(\mathbf{x}) \\ (\partial_0 + \partial_3)^2 D_{2s+1 \ 2s+1}^{(s)\text{flat}}(\mathbf{x}) &= -D_{2s+3 \ 2s+3}^{(s+1)\text{flat}}(\mathbf{x}) \end{aligned} \quad (\text{C.1})$$



follow immediately. From the other hand, in Appendix B we show that the integrand in the r.h.s. of equation (5.1) is a polynomial on  $\omega$ . Then we can bring the limiting passage inside the integral on  $\omega$ . Then using (A.2) we obtain

$$\tilde{D}_{lm}^{(s)\text{flat}}(\mathfrak{x}) = \frac{2^{2s-1}}{2\pi^2(2s+1)} \delta_{lm} \int_{-\infty}^{\infty} d\omega \omega^{2s+1} e^{i\omega(\mathfrak{x}^0 - \mathfrak{x}^3)} {}_1F_1(m, 2s+2, 2i\omega\mathfrak{x}^3).$$

Then the validity of recurrence relations (C.1) for the propagators  $\tilde{D}^{(s)\text{flat}}(\mathfrak{x})$  follows from the formulae [20]

$$\begin{aligned} \frac{d^n}{d\mathfrak{x}^n} {}_1F_1(a, c; \mathfrak{x}) &= \frac{a(a+1) \dots (a+n-1)}{c(c+1) \dots (c+n-1)} {}_1F_1(a+n, c+n; \mathfrak{x}) \\ \frac{d^n}{d\mathfrak{x}^n} \left( e^{-\mathfrak{x}} {}_1F_1(a, c; \mathfrak{x}) \right) &= \frac{(c-a)(c-a+1) \dots (c-a+n-1)}{c(c+1) \dots (c+n-1)} e^{-\mathfrak{x}} {}_1F_1(a, c+n; \mathfrak{x}) \end{aligned}$$

and the formula

$$\frac{1}{a} {}_1F_1(a, a+1; \mathfrak{x}) - \frac{1}{a+1} {}_1F_1(a+1, a+2; \mathfrak{x}) = \frac{1}{a(a+1)} {}_1F_1(a, a+2; \mathfrak{x})$$

which may be easily derived from the relations between the adjacent confluent hypergeometric functions [20].

## References

- [1] Pol'shin S A 2000 *J. Phys. A: Math. Gen.* **33** 5077
- [2] Bytsenko A A, Cognola G, Vanzo L and Zerbini S 1996 *Phys. Rep.* **266** 1
- [3] Pol'shin S A 1999 *Mod. Phys. Lett. A* **14** 2119
- [4] Inagaki T, Ishikawa K and Muta T 1996 *Progr. Theor. Phys.* **96** 847
- [5] Mück W 2000 *J. Phys. A: Math. Gen.* **33** 3021
- [6] Jin W M 2000 *Class. Quantum Grav.* **17** 2949
- Strohmaier A 2000 *Commun. Math. Phys.* **215** 105
- Hollands S 2001 *Commun. Math. Phys.* **216** 635
- [7] Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
- [8] Weinberg S 1964 *Phys. Rev.* **134B** 882
- [9] Naimark M A 1964 *Linear representations of the Lorentz group* (Oxford: Pergamon)
- [10] Verdiyev Yi A 1997 *J. Phys. A: Math. Gen.* **30** 4089
- [11] Stepanovsky Yu P 1981 *Teor. Mat. Fiz.* **47** 343
- [12] Lesimple M 1989 *Lett. Math. Phys.* **18** 315
- [13] Birrell N D and Davies P C W 1982 *Quantum fields in curved space* (Cambridge: Cambridge University Press)
- [14] Schomblond C and Spindel Ph 1976 *Bulletin de l'Académie royale de Belgique (Classe des Sciences), 5eme Série* **LXII** 124
- Gabriel Cl and Spindel Ph 1997 *J.Math.Phys.* **38** 622
- [15] Allen B 1985 *Phys. Rev. D* **32** 3136
- [16] Gazeau J P, Renaud J and Takook M V 2000 *Class. Quantum Grav.* **17** 1415
- [17] Schützhold R 2001 *Phys. Rev. D* **63** 024014
- [18] Bros J and Moschella U 1996 *Rev. Math. Phys.* **8** 327
- Gazeau J P and Takook M V 2000 *J. Math. Phys.* **41** 5920
- [19] Streater R F and Wightman A S 1964 *PCT, Spin and Statistics* (Benjamin, New York)
- [20] Erdélyi A, Magnus W, Oberhettinger F and Tricomi F 1953 *Higher transcendental functions* vol 1 (New York: Mc Graw-Hill)